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NEARLY CIRCULAR TRANSFER TRAJECTORIES FOR DESCENDING SATELLITES

By GEORGE M. LOW

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SUMMARY

Simplified expressions describing the transfer from a satellite orbit to the point of atmospheric entry are derived. The expressions are limited to altitude changes that are small compared with the earth's radius, and velocity changes small compared with satellite velocity. They are further restricted to motion about a spherical, nonrotating earth.

The transfer orbit resulting from the application of thrust in any direction at any point in an elliptic orbit is considered. Expressions for the errors in distance (miss distance) and entry angle due to an initial misalinement and magnitude error of the deflecting thrust are presented.

The largest potential contributing factor towards a miss distance stems from the misalinement of the retrovelocity increment. If this velocity increment is pointed in direct opposition to the flight path, a 1° misalinement leads to a miss distance of 34.5 miles. However, it is shown that this error can be avoided by applying the velocity increment at an angle between 120° and 150° below the flight-path direction.

The guidance and accuracy requirements to establish a circular orbit, in addition to the corrections applied to transform elliptic orbits into circular ones, are also discussed.

INTRODUCTION

The descent trajectories of satellite vehicles can generally be treated in two distinct steps. The first step considers the transfer from the satellite's orbit to a point just outside the earth's atmosphere. The transfer can be initiated by the application of a propulsive force; because this phase of the flight takes place outside the atmosphere, aerodynamic forces are unimportant.

In the second step, the vehicle reenters the atmosphere, so that aerodynamic forces play a very important role in determining the satellite's motion. This phase of the flight of a reentering satellite has been studied in great detail in references 1 and 2.

The present report discusses the transfer trajectory from the satellite orbit to the top of the atmosphere. If the initial satellite orbit is deflected by a single impulsive application of thrust, the transfer trajectory is fully described by Kepler's laws of planetary motion. Although the equations describing these laws are not very complicated, they involve a sufficient number of terms to render it difficult to interpret them without working out a large number of examples.

In general, the transfer trajectory for reentering satellites has two characteristics which permit a great simplification of the equations of motion: The change in altitude is small compared with the distance to the earth's center, and the difference between the actual velocity and local satellite velocity is small.

Equations are derived, making use of these simplifications, which describe the transfer orbit obtained by applying thrust in any direction at any point in an elliptic satellite orbit. With these equations, the time and distance to the point of atmospheric entry and the entry velocity and angle can be calculated in a straightforward manner. Expressions for the errors in distance (miss distance) and entry angle at the top of the earth's atmosphere, caused by an initial misalinement and magnitude error of the deflecting force, are also presented.

In deriving these equations, it is assumed that the earth is nonrotating and spherical. Therefore, the results of this report should not be applied to the computation of an actual reentry and recovery maneuver; instead, they are intended to aid in the rapid estimation of thrust magnitude and accuracy requirements.

The expressions derived for the transfer orbit also permit a rather rapid calculation of the accuracy required to establish a desired satellite orbit. The corrections needed to transform an elliptic orbit to a circular orbit are also given.

SYMBOLS

E	total energy of orbit
g	gravitational constant, $g=g_o[r_o/(r_o+H)]^2$
g_o	gravitational constant at sea level (32.2 ft/sec^2 for earth)
H	altitude above earth's surface
h	angular momentum
N	distance normal to orbital plane
r	distance measured from earth's center
r_o	mean radius of earth (3959 statute miles)
S	lineal distance ($S=r\theta$)
t	time
V	dimensionless velocity ($V=v/v_A^*$)
v	velocity
Δv	velocity increment
v^*	local circular velocity ($v^*=\sqrt{gr}$)
w	fictitious velocity increment for descent from elliptic orbit
α	altitude function referred to circular orbit at point of descent (eq. (13))
β	altitude function referred to apogee of descent orbit (eq. (20))
γ	dimensionless velocity increment ($\gamma=\Delta v/\bar{v}^*$)
δ	angle between original and resulting orbital planes
ϵ	eccentricity of orbit
θ	angle between apogee and point in question
λ	function defined by equation (64)
μ	force constant (1.408×10^{16} ft^3/sec^2 for earth)
ξ	function defined by equation (63)
ρ	radius ratio ($\rho=r/r_A$)
Φ	angle of flight path with respect to local horizontal [1]
ψ	angle of velocity increment with respect to orbital plane
ω	angle of velocity increment with respect to local horizontal,[1] measured in orbital plane

Subscripts:

A	conditions at apogee
max	maximum
P	conditions at perigee
S	conditions in elliptic orbit

Superscripts:

$-$	denotes conditions at the point where the velocity increment is applied
$'$	denotes the actual velocity increment and angle for descent from an elliptic orbit

EQUATION OF MOTION

The path of a satellite follows a Keplerian ellipse (see, e.g., ref. 3) and is described by the following equation:

$$r=\frac{h^2/\mu}{1-\epsilon\cos\theta} \tag{1}$$

where r is the distance from the center of mass, and θ represents the angle measured from the apogee of the ellipse, as shown in figure 1.

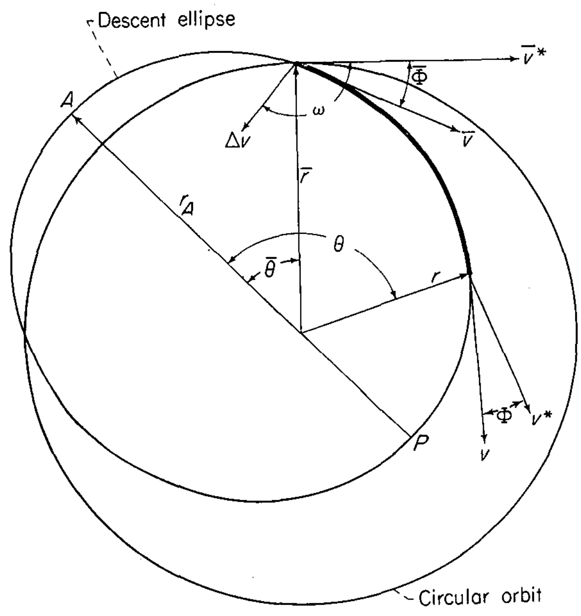


FIGURE 1.—Entry from circular orbit.

[1] The local horizontal is perpendicular to the radius vector, and directed in the plane of the orbit.

The angular momentum h is a constant, given by

$$h = rv \cos \Phi = r_A v_A \tag{2}$$

The force constant μ is related to the gravitational acceleration

$$\mu = gr^2 = r(v^*)^2 \tag{3}$$

where v^* is the local circular orbital velocity. The eccentricity ϵ can be expressed in terms of the total energy E of the orbit

$$\epsilon = \sqrt{1 + \frac{2Eh^2}{\mu^2}} \tag{4}$$

where

$$E = \frac{v^2}{2} - (v^*)^2 \tag{5}$$

Some manipulation with equations (1) to (5) yields expressions for the velocity at any point, the angular distance from the apogee, the local flight-path angle, and the time of flight. The resulting relations may be expressed in terms of a dimensionless velocity, referred to the circular velocity at the apogee

$$V \equiv v/v_A^* \tag{6}$$

and

$$V_A^2 = \left(\frac{v_A}{v_A^*}\right)^2 = 1 - \epsilon \tag{7}$$

and a dimensionless radial distance

$$\rho \equiv r/r_A \tag{8}$$

The final expressions become:
Velocity at any point:

$$V^2 = V_A^2 - 2\left(1 - \frac{1}{\rho}\right) \tag{9}$$

Angular distance from apogee:

$$\cos \theta = \frac{V_A^2 - \rho}{\rho(V_A^2 - 1)} \tag{10}$$

Flight-path angle:

$$\tan \Phi = \sqrt{\frac{V_A^2 - 2}{V_A^2} \rho^2 + \frac{2\rho}{V_A^2} - 1} \tag{11}$$

Time of flight from apogee:

$$\frac{v_A^* t}{r_A} = \frac{1}{(2 - V_A^2)^{3/2}} \cos^{-1}\left[\frac{(2 - V_A^2)\rho - 1}{1 - V_A^2}\right] + \frac{V_A \tan \Phi}{2 - V_A^2} \tag{12}$$

Equations (9) to (12) express the characteristics of the original satellite orbit in a convenient form. They also represent a set of exact equations describing the descent trajectory.

DESCENT FROM CIRCULAR ORBIT

A satellite is presumed to be flying in a circular orbit with a radius $\bar{r}$, at a velocity $\bar{v}^*$ ($v = \bar{v}^*$ for circular orbit). The descent is initiated by applying to the satellite a velocity increment Δv directed at an angle ω measured from the local horizontal in the plane of the orbit. This maneuver will place the satellite on an elliptic orbit, as shown in figure 1. In order to simplify the analysis, it is assumed that: (1) The velocity increment Δv is small compared with the orbital velocity; this assumption implies that the eccentricity of the deflected orbit is small as compared with unity. (2) The change in altitude is small compared with the distance to the center of the earth. A small quantity α is defined

$$\alpha \equiv 1 - \frac{r}{\bar{r}} \tag{13}$$

so that $\alpha << 1$.

As discussed in the appendix, these assumptions imply that several other quantities (i.e., the velocity increment $\Delta v/\bar{v}^*$, and the local flight path angle Φ) are also small when compared with unity. The orbit equations derived in this section are therefore expressed only to the first order; that is, quantities of the order ϵ (eccentricity) are retained, while quantities of higher order of magnitude are neglected. A typical example, given in the appendix, shows that the errors introduced by this linearization are small in a practical case.

ORBIT EQUATIONS

The velocity $\bar{v}$ obtained by deflecting a circular orbit with a velocity increment Δv at an angle ω is obtained from geometric considerations (fig. 1)

$$\bar{v}^2 = (\bar{v}^*)^2 + 2\Delta v \bar{v}^* \cos \omega + (\Delta v)^2 \tag{14}$$

By defining a dimensionless velocity increment γ, where

$$\gamma \equiv \frac{\Delta v}{\bar{v}^*} \tag{15}$$

and dropping terms of order γ^2, the following is obtained:

$$\frac{\bar{v}^2}{(\bar{v}^*)^2} = 1 + 2\gamma \cos \omega \tag{16}$$

The total energy of the orbit, from equations (5) and (16), becomes

$$\frac{2E}{(\bar{v}^*)^2}=-1+2\gamma \cos \omega \tag{17}$$

and the angular momentum is

$$h=\bar{r}\bar{v}^*(1+\gamma \cos \omega) \tag{18}$$

As shown in the appendix (eq. (A8)), the eccentricity may be expressed as

$$\epsilon=\gamma\sqrt{1+3 \cos^2\omega} \tag{19}$$

It is convenient to refer the local distance to the earth's center r to the radius r_A at the apogee of the descent trajectory rather than to the radius $\bar{r}$ of the original orbit. The new radius ratio is defined in terms of

$$\beta\equiv 1-r/r_A \tag{20}$$

or

$$\bar{\beta}=1-(\bar{r}/r_A) \tag{21}$$

so that, to the first order

$$\beta=\alpha+\bar{\beta} \tag{22}$$

From equation (2)

$$\frac{r_A}{\bar{r}}=\frac{\bar{v}}{v_A} \cos \bar{\Phi}$$

From use of the fact that

$$\left(\frac{\bar{v}^*}{v_A^*}\right)^2=\frac{r_A}{\bar{r}} \tag{23}$$

there results

$$\bar{\beta}=\gamma\left(2 \cos \omega+\sqrt{1+3 \cos^2\omega}\right) \tag{24}$$

and

$$\beta=\alpha+\gamma\left(2 \cos \omega+\sqrt{1+3 \cos^2\omega}\right) \tag{25}$$

With these expressions it is possible to express the desired quantities in terms of α (representing the altitude), γ (representing the velocity increment), and ω (representing the direction of the applied velocity increment).

Velocity at any point.—The local velocity in the flight-path direction is obtained from equations (9) and (23):

$$\begin{aligned}\frac{v^2}{(\bar{v}^*)^2}&=\frac{(v_A^*)^2}{(\bar{v}^*)^2}\frac{v^2}{(v_A^*)^2}\\&=(1-\bar{\beta})\left[(1-\epsilon)-2\left(1-\frac{1}{1-\beta}\right)\right]\\&=1-\bar{\beta}-\epsilon+2\beta\end{aligned} \tag{26}$$

In terms of the known quantities α, γ, and ω, the local velocity, referred to the circular velocity of the original orbit, becomes

$$\frac{v^2}{(\bar{v}^*)^2}=1+2\alpha+2\gamma \cos \omega \tag{27}$$

Distance traveled.—The distance from the point of thrust application to an arbitrary point is expressed in terms of the angular distance θ from the apogee of the entry orbit and the apogee shift $\bar{\theta}$ measured from the same apogee to the point or orbit deflection. From equation (10),

$$\cos \theta=\frac{(1-\epsilon)-(1-\beta)}{(1-\beta)(-\epsilon)}=\left(1-\frac{\beta}{\epsilon}\right)(1+\beta) \tag{28}$$

In terms of the given quantities α, γ, and ω equation (28) becomes:

$$\cos \theta=-\frac{2 \cos \omega+(\alpha/\gamma)}{\sqrt{1+3 \cos^2\omega}}\left[1+\alpha+\gamma\left(2 \cos \omega+\sqrt{1+3 \cos^2\omega}\right)\right] \tag{29}$$

The apogee shift $\bar{\theta}$ is derived by similar means:

$$\cos \bar{\theta}=-\frac{2 \cos \omega}{\sqrt{1+3 \cos^2\omega}}\left[1+\gamma\left(2 \cos \omega+\sqrt{1+3 \cos^2\omega}\right)\right] \tag{30}$$

The actual angular distance traveled is, from figure 1, equal to $\theta\mp\bar{\theta}$; here the minus sign applies when $\omega<180°$, while the plus sign applies when $\omega>180°$. The lineal distance is given by

$$\begin{aligned}S&=r(\theta-\bar{\theta}) \quad \omega<180°\\S&=r(\theta+\bar{\theta}) \quad \omega>180°\end{aligned} \tag{31}$$

Flight-path angle.—The local inclination of the flight path is given by equation (11). Replacing V_A^2 by $1-\epsilon$, and ρ by $1-\beta$, yields the expression

$$\tan \Phi\cong\Phi=\beta\sqrt{\frac{2\epsilon}{\beta}-1} \tag{32}$$

In terms of α, γ, and ω equation (32) becomes

$$\Phi=\sqrt{\gamma^2 \sin^2\omega-\alpha^2-4\alpha\gamma \cos \omega} \tag{33}$$

For some applications it may be desired to determine the velocity increment (or γ) required to achieve a given angle Φ at a certain altitude (or α). Inversion of equation (33) yields

$$\gamma=\alpha\left\{\frac{2 \cos \omega+\sqrt{1+3 \cos \omega+\frac{\Phi^2}{\alpha^2} \sin^2\omega}}{\sin^2\omega}\right\} \tag{34}$$

In the limit, as ω approaches 180°, equation (34) becomes

$$\gamma=\frac{\alpha}{4}\left[1+\frac{\Phi^2}{\alpha^2}\right] \tag{35}$$

Time of flight.—The time of flight from the point of orbit deflection is the sum (or difference) of the time from apogee, and the time of travel to the shifted apogee. From equation (12)

$$\frac{\bar{v}^*t}{\bar{r}}=\left[1+\frac{3}{2}(\bar{\beta}-\epsilon)\right]\left[\cos^{-1}\left(1-\frac{\beta}{\epsilon}\right)\mp\cos^{-1}\left(1-\frac{\bar{\beta}}{\epsilon}\right)\right]+(\Phi\mp\bar{\Phi}) \tag{36}$$

where the minus sign applies for $\omega<180°$, and the plus sign applies for $\omega>180°$. In terms of the known quantities, equation (36) becomes

$$\frac{\bar{v}^*t}{\bar{r}}=(1+3\gamma\cos\omega)\left[\cos^{-1}\left(-\frac{\frac{\alpha}{\gamma}+2\cos\omega}{\sqrt{1+3\cos^2\omega}}\right)\mp\cos^{-1}\left(\frac{-2\cos\omega}{\sqrt{1+3\cos^2\omega}}\right)\right]+[\sqrt{\gamma^2\sin^2\omega-\alpha^2-4\alpha\gamma\cos\omega}\mp\gamma\sin\omega] \tag{37}$$

The expressions for velocity, flight distance, entry angle, and flight time are listed in table I; also shown in table I are the simplified expressions for $\omega=90°$, 180°, and 270°.

REENTRY ERRORS

If a successful recovery of a satellite is to be achieved, it is of utmost importance to minimize the miss distance resulting from small deviations in the magnitude or direction of the applied velocity increment. The simplified orbit equations of the preceding section are particularly suited to obtain expressions for these orbit deviations.

Miss distance.—The miss distance caused by an error in the magnitude of Δv is obtained by differentiation of equations (29) and (30) with respect to γ, and subsequent substitution into the derivative of equation (31). The resulting expression is

$$\frac{1}{r}\frac{\partial S}{\partial\gamma}=\frac{-\alpha}{\Phi\gamma} \tag{38}$$

or

$$\frac{\partial S}{\partial(\Delta v)}=-\frac{r}{\bar{v}^*}\frac{\alpha}{\Phi\gamma} \tag{39}$$

A misalinement of the retrothrust in the plane of the orbit causes the following miss distance:

$$\frac{\partial S}{\partial\omega}=\frac{2r}{1+3\cos^2\omega}\left[\left(\frac{3}{2}\alpha\cos\omega-\gamma\right)\frac{\sin\omega}{\Phi}+1\right] \tag{40}$$

where the plus sign (in the brackets) was selected on the basis of physical reasoning. This miss distance, caused by retrothrust misalinement in the orbital plane, is the sum (or difference) of two errors; the first is a change in the location of the apogee of the descent trajectory, whereas the second is a change in the distance traveled from the shifted apogee. In general, it is possible to orient the velocity increment so that these two errors cancel each other. From equation (40), the miss distance $\partial S/\partial\omega$ vanishes when

$$\frac{\gamma}{\alpha}=\frac{\frac{9}{4}\sin^2\omega\cos^2\omega+1}{\cos\omega\,(3\sin^2\omega-4)} \tag{41}$$

The corresponding expression for entry angle is (from eqs. (33) and (41))

$$\frac{\Phi}{\alpha}=\tan\omega\left[\frac{1-\frac{3}{4}\cos^2\omega\,(3\sin^2\omega-8)}{3\sin^2\omega-4}\right] \tag{42}$$

The values of γ/α and Φ/α for $\partial S/\partial\omega=0$ are plotted in figure 2.

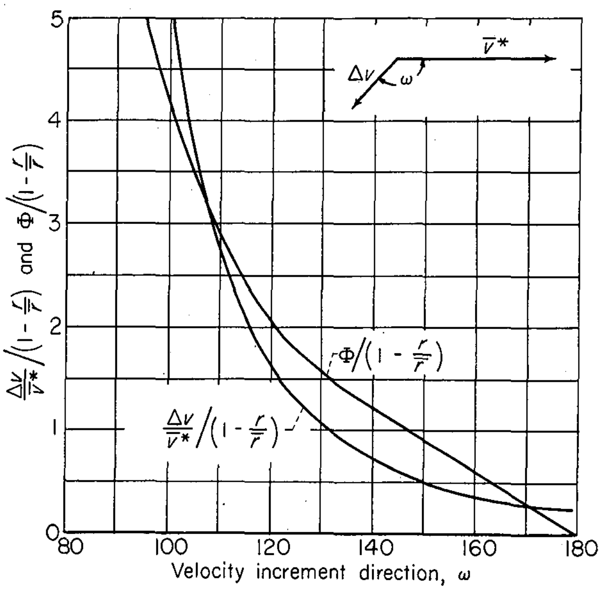


FIGURE 2.—Velocity increment and entry angle for zero miss distance $\partial S/\partial\omega$.

If the velocity increment should be directed out of the plane of the orbit, the plane of the descent trajectory will differ from the orbital plane. The point of entry into the earth's atmosphere will then be displaced by an azimuthal distance N, measured normal to the orbital plane. We assume that the velocity increment is directed at an angle ψ out of the orbital plane; as indicated in figure 3,

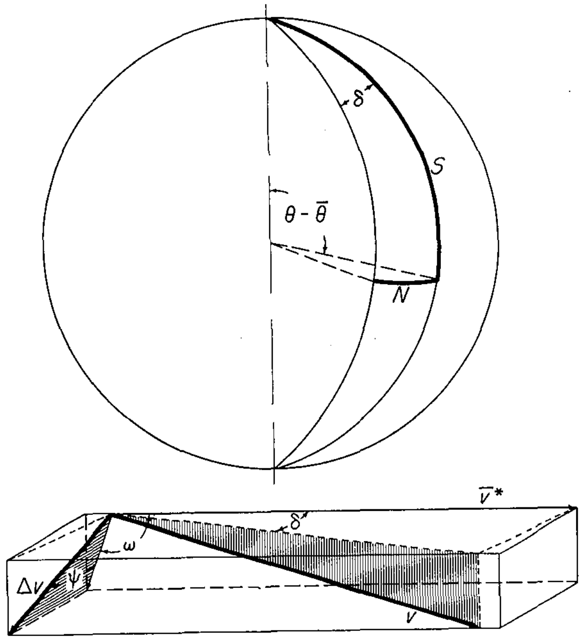


FIGURE 3.—Velocity increment directed out of orbital plane.

ψ is measured in a plane passing through Δv and perpendicular to the orbital plane. The resulting trajectory will be in a plane making an angle δ with the original plane, where, from the geometry of figure 3,

$$\delta \cong \gamma \sin \psi$$

The magnitude of the normal distance N depends on the altitude (or r) and the distance traveled $(\theta \mp \bar{\theta})$:

$$N = r\gamma \sin \psi \sin (\theta \mp \bar{\theta})$$

The miss distance, for small angles ψ, becomes

$$\frac{\partial N}{\partial \psi} = r\gamma \sin (\theta \mp \bar{\theta}) \tag{43}$$

Therefore, the normal miss distance maximizes for a 90° entry range, and vanishes for a 180° entry range.

Expressions for the miss distance resulting from simultaneous errors in γ, ω, and ψ were not derived. However, it is intuitively clear that each individual miss distance will maximize when the other miss distances are zero. Therefore equations (39), (40), and (43) represent the maximum range and azimuth miss distances at the top of the atmosphere.

Entry-angle error.—Unfortunately, the miss distance computed with the aid of equations (39) and (40) is only one part of the total range miss distance. The initial thrust error also causes an error of the atmospheric entry angle (eq. (33)) this entry-angle error in turn will alter the distance traveled within the earth's atmosphere. The miss distance incurred while flying within the atmosphere can be determined by the methods of reference 1, once the entry-angle error is known. The entry-angle error resulting from an error in the magnitude of Δv is, from equation (33),

$$\frac{\partial \Phi}{\partial \gamma} = \frac{\gamma}{\Phi}\left(\sin^2 \omega - \frac{2\alpha}{\gamma} \cos \omega\right) \tag{44}$$

or

$$\frac{\partial \Phi}{\partial \Delta v} = \frac{\gamma}{\Phi \bar{v}^*}\left(\sin^2 \omega - \frac{2\alpha}{\gamma} \cos \omega\right) \tag{45}$$

Thrust axis misalinement in the orbital plane causes the following error

$$\frac{\partial \Phi}{\partial \omega} = \frac{\gamma^2 \sin \omega}{\Phi}\left(\cos \omega + 2\,\frac{\alpha}{\gamma}\right) \tag{46}$$

This error can be made to vanish by directing the thrust axis so that

$$\cos \omega = -2\,\frac{\alpha}{\gamma} \tag{47}$$

The entry angle calculated by the method of this report becomes an initial condition for the calculations of atmospheric flight by the method of reference 1. Generally, the solutions for extra-atmospheric flight and intra-atmospheric flight are matched at an altitude between 50 and 70 miles. The question naturally arises as to how critical the matching altitude is, or how much the entry angle changes over a narrow band of altitudes. This change in angle is

$$\frac{\partial \Phi}{\partial r} = \frac{\gamma}{r\Phi}\left(2 \cos \omega + \frac{\alpha}{\gamma}\right) \tag{48}$$

Time of flight error.—When calculating the actual miss distance for a rotating earth, the distance that a point on the earth's surface has moved during the time interval between the predicted and actual arrival of the vehicle must be considered. Because this distance depends on the type and direction of the orbit, only the time error will be presented herein. An error in the magnitude of the velocity increment causes the following time error (from eq. (37)):

$$\frac{\bar{v}^*}{\bar{r}}\frac{\partial t}{\partial \gamma}=-\frac{\alpha}{\Phi\gamma} \tag{49}$$

The time error resulting from a thrust misalinement is

$$\frac{\bar{v}^*}{\bar{r}}\frac{\partial t}{\partial \omega}=\frac{2}{1+3\cos^2\omega}\left[\left(\frac{3}{2}\alpha\cos\omega-\gamma\right)\frac{\sin\omega}{\Phi}+1\right] \tag{50}$$

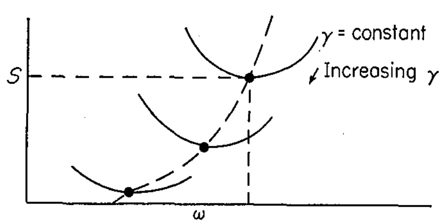
Note that the nondimensional time-error coefficients are identical to the nondimensional range errors, as given by equations (39) and (40). Therefore, if the satellite is travelling in the direction of the earth's rotation, the actual miss distance will be reduced by the time of flight error.

A summary of the various error equations is given in table II.

MINIMIZATION OF REQUIRED IMPULSE

The impulse required to execute the reentry maneuver can generally be minimized by orienting the velocity increment in a predetermined direction. This direction depends on whether a given entry range or a given entry angle is called for.

Minimum impulse for fixed range.—If the entry range (S) is fixed, the thrust orientation for minimum impulse coincides with that for vanishing miss distance caused by thrust misalinement (i.e., $\partial S/\partial\omega=0$). This fact can best be demonstrated by examining the following sketch:



The solid lines in the sketch represent lines of constant γ, and increasing values of γ correspond to decreasing values of the range S. For any given range, therefore, the minimum value of the velocity increment γ corresponds to the value of ω for $\partial S/\partial\omega=0$, as given by equation (41). A relation between angular range and velocity increment orientation is obtained by substituting equation (41) into equations (29) and (30). In general, this expression is a function of α (representing the altitude). However, to zero order, a universal relation for the velocity-increment orientation (ω) for minimum impulse for a fixed angular range is obtained. This relation, which is plotted in figure 4, is

$$\theta\mp\bar{\theta}=\cos^{-1}\left[\frac{-2\cos\omega}{\sqrt{1+3\cos^2\omega}}\left(1-\frac{1+3\cos^2\omega}{\frac{9}{2}\sin^2\omega\cos^2\omega+2}\right)\right]$$

$$\mp\cos^{-1}\left(\frac{-2\cos\omega}{\sqrt{1+3\cos^2\omega}}\right) \tag{51}$$

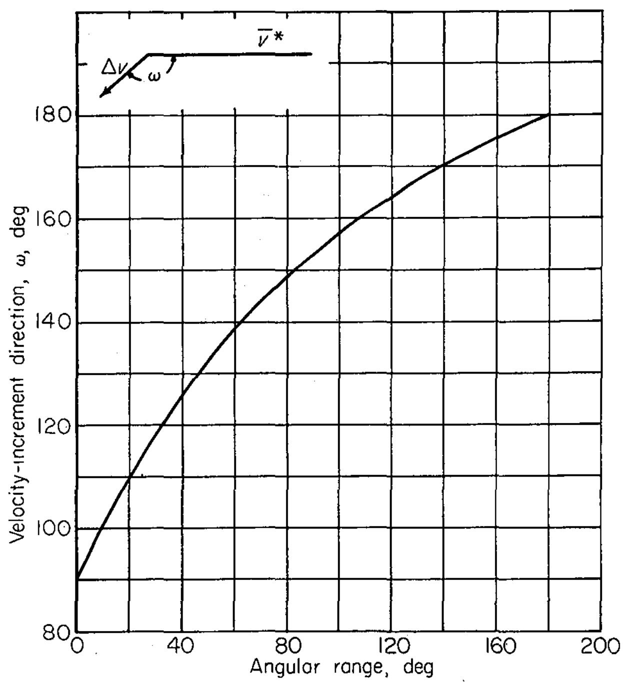


FIGURE 4.—Velocity-increment direction for minimum impulse for fixed range.

Minimum impulse for fixed entry angle.—The minimum impulse for a fixed entry angle occurs at the value of ω corresponding to a vanishing error $\partial\Phi/\partial\omega$, that is, $\cos\omega=-2\alpha/\gamma$ (eq. (47)).

The relation between ω and Φ is

$$\left.\begin{array}{ll}\cos\omega=\dfrac{-2}{\sqrt{\dfrac{\Phi^2}{\alpha^2}-3}} & \dfrac{\Phi^2}{\alpha^2}\geqq 7\\[2ex] \omega=180^\circ & \dfrac{\Phi^2}{\alpha^2}\leqq 7\end{array}\right\} \tag{52}$$

In other words, the best impulse direction (minimum for fixed entry angle) is 180° for small values of entry angle. The relation of ω as a function of Φ/α is plotted in figure 5.

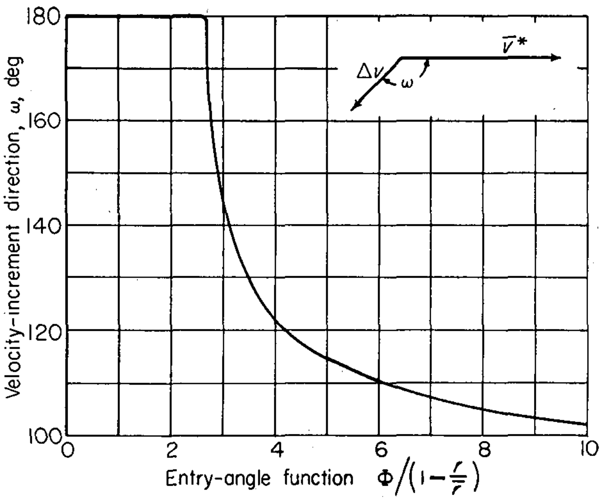


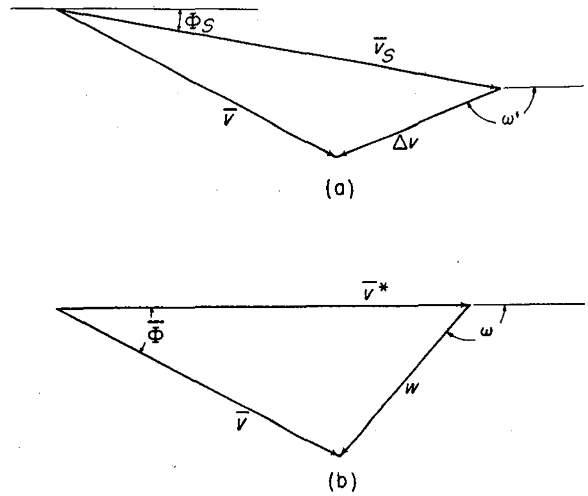
FIGURE 5.—Velocity-increment direction for minimum impulse for fixed entry angle.

DESCENT FROM ELLIPTIC ORBIT

The expressions derived for the descent from a circular orbit can be adapted to elliptic orbits by replacing the actual velocity increment Δv at angle ω' by a fictitious increment w at angle ω.

In figure 6, $\bar{v}^*$ represents the circular velocity at the point of orbit deflection, v_S the satellite velocity in its elliptic orbit, and $\bar{v}$ the resulting velocity after an increment Δv is applied; ω' represents the direction of Δv below the local horizontal. The resulting velocity $\bar{v}$ could also have been obtained by deflecting a circular orbit with a velocity increment w at an angle ω.

With the assumption that conditions at the apogee of the original elliptic orbit are known, the velocity $\bar{v}_S$ and the angle $\bar{\Phi}$ can be obtained from equations (9) and (11). The corresponding distance from the apogee and time of flight are obtained from equations (10) and (12). From the geometry of figure 6



(a) Velocity increment applied to elliptic orbit.

(b) Fictitious velocity applied to circular orbit yielding same final velocity as in figure 6(a).

FIGURE 6.—Entry from elliptic orbit.

$$\left(\frac{\bar{v}}{\bar{v}^*}\right)^2=\left(\frac{\bar{v}_S}{\bar{v}^*}\right)^2+(\gamma')^2+2\,\frac{\bar{v}_S}{\bar{v}^*}\,\gamma'\cos(\omega'-\Phi_S) \tag{53}$$

$$\bar{\Phi}=\Phi_S+\cos^{-1}\left[\frac{\left(\dfrac{\bar{v}_S}{\bar{v}^*}\right)^2+\left(\dfrac{\bar{v}}{\bar{v}^*}\right)^2-(\gamma')^2}{2\,\dfrac{\bar{v}_S\bar{v}}{(\bar{v}^*)^2}}\right] \tag{54}$$

$$\gamma=\left[\left(\frac{\bar{v}}{\bar{v}^*}\right)^2+1-2\,\frac{\bar{v}}{\bar{v}^*}\cos\bar{\Phi}\right]^{1/2} \tag{55}$$

$$\omega=\cos^{-1}\left[\frac{\left(\dfrac{\bar{v}}{\bar{v}^*}\right)^2-1-\gamma^2}{2\gamma}\right] \tag{56}$$

where

$$\gamma\equiv w/\bar{v}^*;\qquad \gamma'\equiv\Delta v/\bar{v}^*$$

The following procedure is to be followed for satellite descent from an elliptic orbit:

(1) Obtain $\bar{v}_S$ at the desired r from equation (9); find Φ_S from equation (11), and $\bar{v}^*$ from equation (3).

(2) For a given Δv (inclined at ω'), find $\gamma'=\Delta v/\bar{v}^*$.

(3) Then, equations (53) and (54) yield $\bar{v}/\bar{v}^*$ and Φ.

(4) Calculate γ (eq. (55)) and ω (eq. (56)).

(5) Use the values of γ and ω so obtained directly in the expressions of table I to compute conditions along the descent orbit.

The miss distance and entry-angle error caused by an error in either γ' or ω' can be expressed in terms of the derivatives in table II, together with the following expressions:

$$\frac{\partial}{\partial\omega'}=\frac{\partial\omega}{\partial\omega'}\frac{\partial}{\partial\omega}+\frac{\partial\gamma}{\partial\omega'}\frac{\partial}{\partial\gamma} \tag{57}$$

$$\frac{\partial}{\partial\gamma'}=\frac{\partial\omega}{\partial\gamma'}\frac{\partial}{\partial\omega}+\frac{\partial\gamma}{\partial\gamma'}\frac{\partial}{\partial\gamma} \tag{58}$$

The derivatives of ω and γ with respect to ω' and γ' are obtained from equations (53) to (56); the results are:

$$\frac{\partial\gamma}{\partial\gamma'}=\frac{\gamma'+\xi\cos(\lambda-\omega')}{\gamma} \tag{59}$$

$$\frac{\partial\gamma}{\partial\omega'}=\frac{\gamma'\xi\sin(\lambda-\omega')}{\gamma} \tag{60}$$

$$\frac{\partial\omega}{\partial\gamma'}=\frac{-\xi\sin(\lambda-\omega')}{\gamma^2} \tag{61}$$

$$\frac{\partial\omega}{\partial\omega'}=\frac{\gamma'[\gamma'+\xi\cos(\lambda-\omega')]}{\gamma^2} \tag{62}$$

where

$$\xi\equiv\sqrt{1+\left(\frac{\bar{v}_S}{\bar{v}^*}\right)^2-2\frac{\bar{v}_S}{\bar{v}^*}\cos\Phi_S} \tag{63}$$

and

$$\lambda\equiv\Phi_S+\sin^{-1}\left(\frac{\sin\Phi_S}{\xi}\right) \tag{64}$$

A small eccentricity of the initial elliptic satellite orbit is not, in general, necessary for this analysis to apply. However, the descent orbit must again be nearly circular. If the eccentricity of the satellite orbit is large, then a relatively large velocity increment Δv is required to achieve a nearly circular descent trajectory. In a practical case, it may be desirable to apply a smaller velocity increment and to descend on a path of greater eccentricity. This latter case cannot be treated by the present analysis.

DISCUSSION OF DESCENT TRAJECTORIES

ENTRY FROM 150-MILE ORBIT

For the sake of simplifying the discussion of the equations, an example was worked out. For this example, an initial circular orbit at an altitude of 150 statute miles was selected; entry into the atmosphere is accomplished at an altitude of 50 miles. Conditions that are computed at the 50-mile altitude could be matched with solutions valid within the atmosphere, such as those discussed in reference 1.

Orbit characteristics.—The range (in degrees of arc) traveled from the 150-mile altitude to the 50-mile altitude is shown in figure 7 as a function

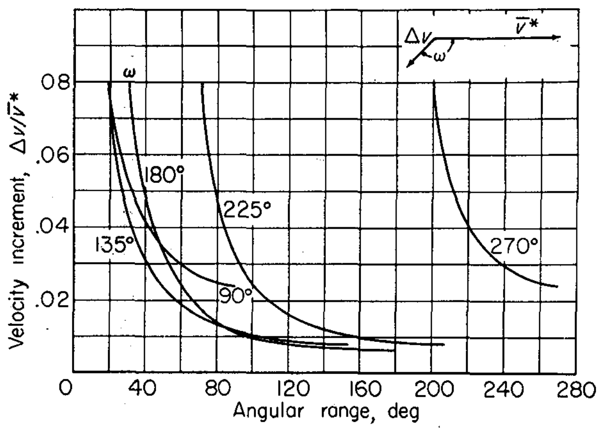


FIGURE 7.—Range as a function of velocity increment. Circular-orbit altitude, 150 statute miles; entry altitude, 50 statute miles.

of the velocity increment $\Delta v/\bar{v}^*$. As previously shown in figure 4, the required thrust is minimized when the velocity increment is applied at an angle between 90° and 180° for angular ranges less than 180°. However, for ranges larger than 90° there is only a small difference in the required thrust when ω lies between 135° and 180°. Very long ranges are possible by directing the velocity increment at angles larger than 180°.

The corresponding entry angles are shown in figure 8. Reference 1 shows that the maximum deceleration during the entry of nonlifting vehicles becomes excessively large for entry angles larger than about 3°; therefore, the maximum allowable angle for manned entry (with a nonlifting vehicle) is of the order of 3°. Again, as discussed with the aid of figure 5, the proper thrust vector orientation for minimum impulse depends on the entry angle. But, for entry angles less than 4½° the penalty paid for applying Δv at 135° rather than at 180° is not large.

Range as a function of entry angle is presented in figure 9. For entry angles of practical interest (between 2° and 3°) and for values of $\omega=180°$ or

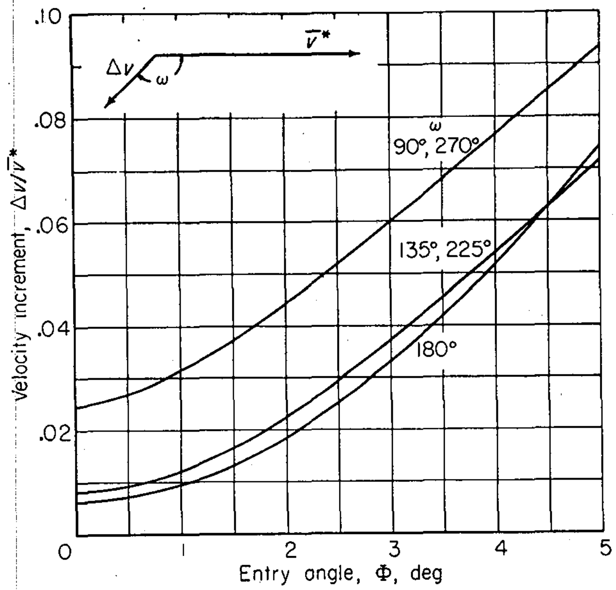


FIGURE 8.—Entry angle for descent from 150-mile circular orbit. Entry altitude, 50 statute miles.

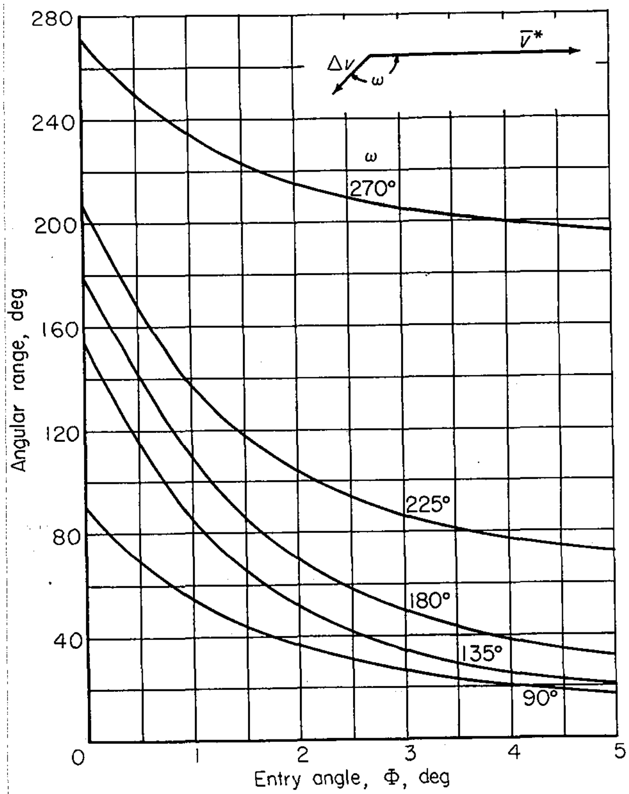


FIGURE 9.—Range as a function of entry angle for descent from 150-statute-mile circular orbit; entry at 50-statute-mile altitude.

less, the distance traveled along the descent trajectory is less than one-fourth of the earth's circumference.

Effect of thrust misalinement.—The error coefficient $\partial S/\partial\omega$ caused by an initial misalinement (in the orbital plane) of the velocity increment is plotted in figure 10 as a function of entry

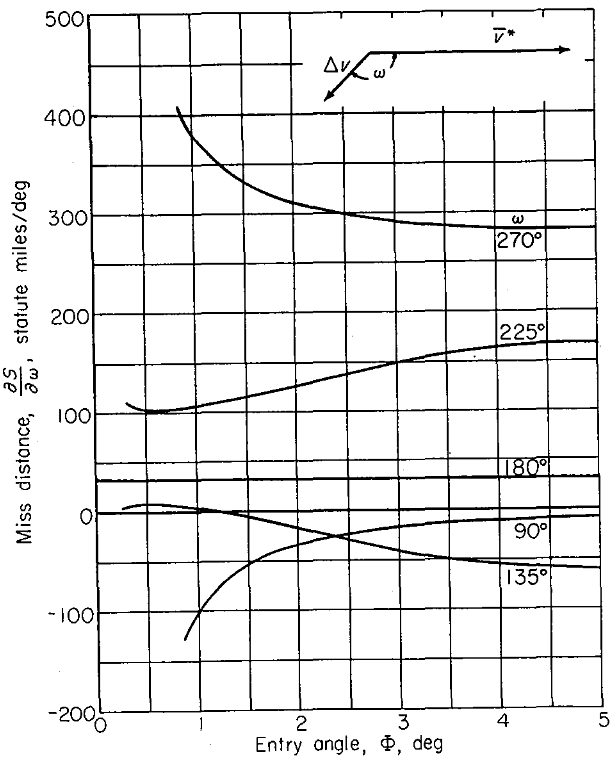


FIGURE 10.—Miss distance caused by thrust misalinement. Orbit altitude, 150 statute miles; entry altitude, 50 statute miles.

angle. For Δv directed opposite to the flight-path direction ($\omega=180°$), the miss distance is constant at a value of 34.5 miles per degree. This miss distance stems from a shift in the apogee of the descent trajectory caused by thrust misalinement. (This quantity was checked against the complete equations (eqs. (9) to (11)); the results of the exact equations agreed to within ½ percent with the present analysis.) The aforementioned apogee shift can be partially or completely cancelled by the change in distance traveled from the shifted apogee when Δv is applied in the proper direction (see fig. 2). The velocity-increment orientation for vanishing $\partial S/\partial\omega$ is shown in figure 11 as a function of entry angle.

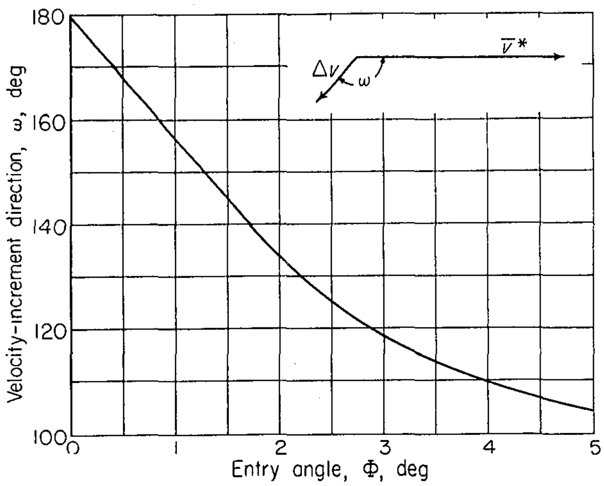


FIGURE 11.—Velocity-increment direction for zero miss distance $\partial S/\partial \omega$. Circular orbit altitude, 150 statute miles; entry altitude, 50 statute miles.

Values of ω between 120° and 135° yield entry angles of practical interest. As discussed in the section Minimization of Required Impulse, these same values of ω yield the minimum impulse for a fixed range.

When ω is greater than 180°, the miss distance becomes prohibitively large (fig. 10). Consequently, the large entry ranges that can be achieved for $\omega > 180°$ are probably ruled out. Very small entry angles ($<0.5°$) are also not recommended, because they lead to very large errors (see table II).

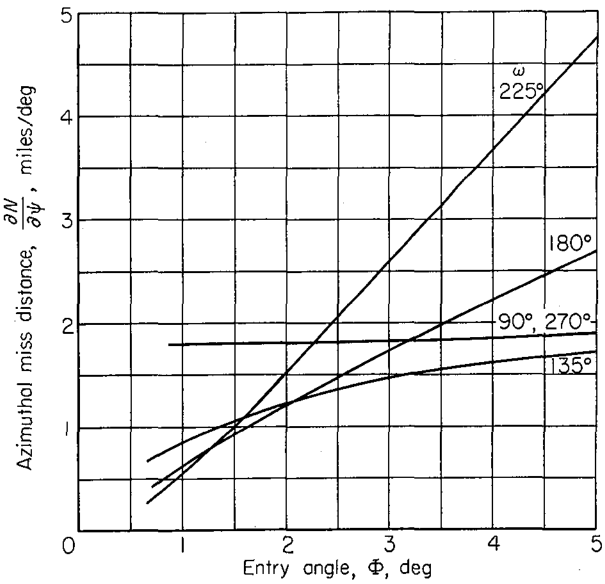


FIGURE 12.—Azimuthal miss distance for descent from 150-mile circular orbit. Entry altitude, 50 statute miles.

The azimuthal miss distance ($\partial N/\partial \psi$) for the example under consideration is shown in figure 12. For practical cases of manned reentry with nonlifting vehicles, this miss distance is generally less than 2 miles per degree. Therefore, thrust misalinement normal to the orbital plane is much less serious than an equal amount of misalinement within the plane of the orbit.

The error in entry angle resulting from thrust misalinement is shown in figure 13. This error

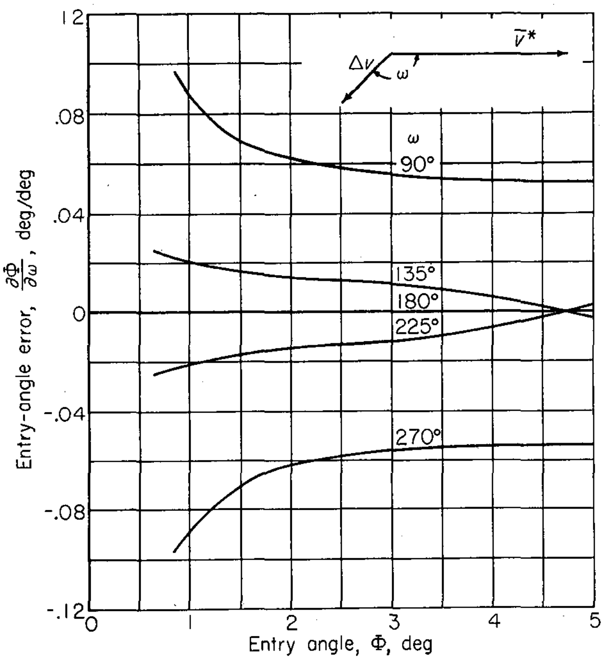


FIGURE 13.—Entry-angle error caused by thrust misalinement. Orbit altitude, 150 statute miles; entry altitude, 50 statute miles.

is zero for $\omega=180°$, and small for $\omega=135°$ and 225°. As far as the deceleration and heating are concerned, even a 0.5° entry-angle error may not cause catastrophic results. For example, reference 1 indicates that a deceleration value of 9.1 g's at $\Phi=2°$ is increased to 9.9 g's at $\Phi=2.5°$. The results of figure 13 indicate that even a rather large thrust misalinement of 10° will only alter the entry angle by 0.14° (for $\Phi=2°$, $\omega=135°$).

The change in entry angle also causes a miss distance which must be added to the other errors. Reference 1 indicates that the entry-angle error of 0.014° per degree of thrust axis misalinement (for $\Phi=2°$, $\omega=135°$) results in a miss distance of 3.36

miles per degree of thrust misalinement. Although this error is by no means negligible, it is an order of magnitude smaller than the value of $\partial S/\partial\omega$ obtained for $\omega=180°$.

Effect of thrust magnitude error.—The miss distance $\partial S/\partial\Delta v$ caused by an error in the velocity increment is shown in figure 14. This miss distance can be minimized by using the largest possible entry angle. For example, the error can be decreased from 15 miles per foot per second to 5 miles per foot per second by increasing the entry angle from 1° to 2° when $\omega=135°$; an additional decrease to 2 miles per foot per second results from an increase of entry angle to 3°.

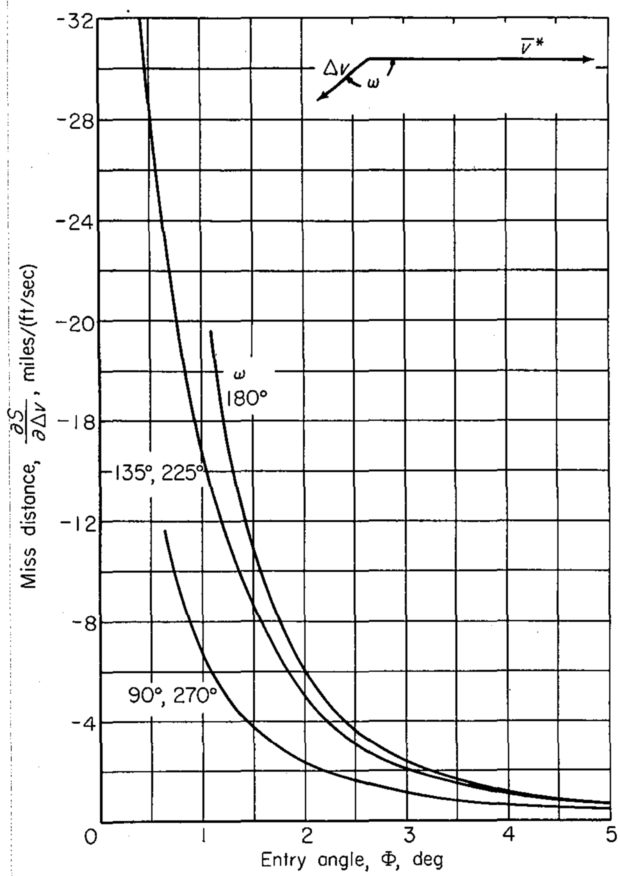


FIGURE 14.—Miss distance caused by thrust magnitude error. Orbit altitude, 150 statute miles; entry altitude, 50 statute miles.

The entry-angle error caused by a thrust magnitude error also decreases rapidly with increasing entry angle (fig. 15).

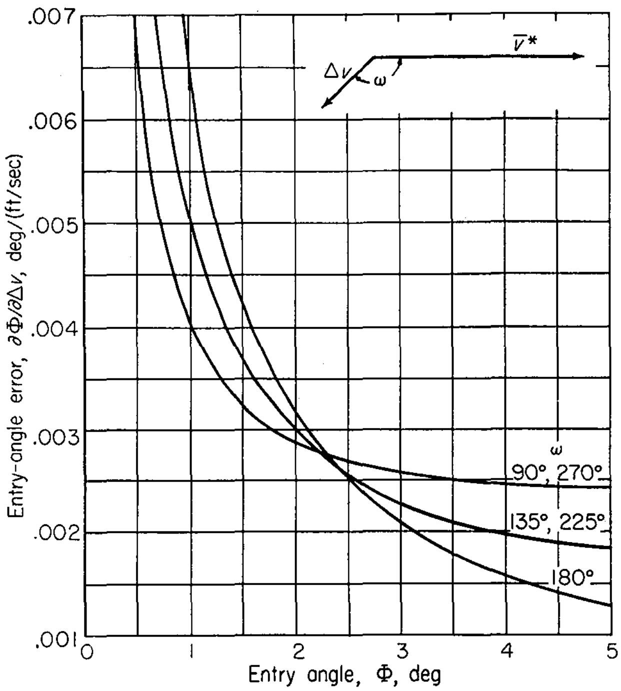


FIGURE 15.—Entry-angle error caused by thrust magnitude error. Orbit altitude, 150 statute miles; entry altitude, 50 statute miles.

Effect of altitude on entry angle.—As mentioned earlier, the entry angle obtained from the present analysis must be matched at a given altitude with the solutions for flight within the atmosphere. Generally, the matching altitude is taken between 50 and 70 miles; all example calculations were made for descent to a 50-mile altitude. An examination of equation (48) indicates that the exact matching altitude is relatively unimportant unless Φ becomes vanishingly small. For the example calculations, with $\Phi=2°$ and $\omega=135°$, $\partial\Phi/\partial r=0.003°$ per mile. In other words, the entry angle changes only 0.03° over a 10-mile change in altitude.

OPTIMUM DESCENT CONDITIONS

Although the example developed for the study of descent trajectories considered only a single orbit altitude, some generalizations can be made by considering the equations of tables I and II.

First it should be noted (from eqs. (9) to (11)) that there is not a great deal of freedom in selecting the descent trajectory once the satellite altitude, the entry altitude, and the entry angle are specified. In fact, if in addition the thrust direction is also specified, then only one descent trajectory exists, and there is no choice of velocity increment, entry range, or velocity. In general,

however, the designer can select any one of a number of combinations of values of Δv and ω.

If the descent of a nonlifting (and uncontrolled) vehicle is considered, then the miss distance may well be the dominating factor in the selection of a reentry system. From the equations of table II, it is clear that large entry angles tend to minimize the miss distance. However, if the vehicle is to be manned, then the entry angle is limited to about 3° in order to avoid intolerable decelerations. Hence, a logical choice of entry angle is between 2° and 3°.

The desired entry angle can be achieved with the velocity increment oriented in any direction between 90° and 270°. For a fixed small entry angle the thrust is minimized for $\omega=180°$, with only a small increase up to 135° or 225°; the entry range is smallest for $\omega=90°$ and increases for larger values of ω. However, unless the miss distance is unimportant, the thrust vector orientation cannot be selected on the basis of minimum thrust or entry range. The miss distance caused by thrust axis misalinement (in the plane of the orbit) can become exceedingly large unless Δv is directed between 90° and 180°. The best orientation depends on the entry angle and orbit altitude, but, in general, will fall between 120° and 150°. This range of angles also represents a good compromise for minimizing all other errors.

ASCENT TO CIRCULAR ORBIT

The present analysis also lends itself to the calculation of guidance and accuracy requirements for the establishment of a circular satellite orbit The desired orbit altitude is assumed to be $H=\bar{r}-r_o$, with a corresponding circular velocity of $\bar{v}^*=\sqrt{g\bar{r}}$. The actual velocity attained by the launching system at altitude H is $\bar{v}=v^*+\Delta v$, and is inclined at an angle $\bar{\Phi}$ below the local horizontal. The effect of the velocity error Δv and angle error $\bar{\Phi}$ on the resulting orbit is obtained as follows:

Equation (9) shows that

$$\frac{\bar{v}^2}{(v_A^*)^2}=\frac{v_A^2}{(v_A^*)^2}-2\left(1-\frac{r_A}{\bar{r}}\right)$$

or

$$\frac{\bar{v}^2}{(\bar{v}^*)^2}=\frac{v_A^2}{(\bar{v}_A^*)^2}\frac{(v_A^*)^2}{(\bar{v}^*)^2}-2\frac{(v_A^*)^2}{(\bar{v}^*)^2}\left(1-\frac{r_A}{\bar{r}}\right) \tag{65}$$

But

$$\frac{v_A^2}{(v_A^*)^2}=1-\epsilon$$

and

$$\frac{(v_A^*)^2}{(\bar{v}^*)^2}=\frac{\bar{r}}{r_A}=1-\bar{\beta}$$

Therefore,

$$\frac{\bar{v}^2}{(\bar{v}^*)^2}=1+2\frac{\Delta v}{\bar{v}^*}=1+\bar{\beta}-\epsilon \tag{66}$$

From equation (32)

$$\bar{\beta}=\epsilon-\sqrt{\epsilon^2-\bar{\Phi}^2} \tag{67}$$

so that the eccentricity of the satellite orbit is

$$\epsilon=\sqrt{\bar{\Phi}^2+4\left(\frac{\Delta v}{\bar{v}^*}\right)^2} \tag{68}$$

The apogee height is represented by

$$\bar{\beta}=1-\frac{\bar{r}}{r_A}=\sqrt{\bar{\Phi}^2+4\left(\frac{\Delta v}{\bar{v}^*}\right)^2}-\frac{2\Delta v}{\bar{v}^*} \tag{69}$$

The maximum deviation from a circular orbit is often of interest. In dimensionless form

$$\frac{(\Delta r)_{max}}{\bar{r}}=\frac{\bar{r}-r_P}{\bar{r}}=2\epsilon-\bar{\beta}=\sqrt{\bar{\Phi}^2+4\left(\frac{\Delta v}{\bar{v}^*}\right)^2}+\frac{2\Delta v}{\bar{v}^*} \tag{70}$$

Figure 16 is a summary plot of the effect of velocity and angle error on orbit deviation.

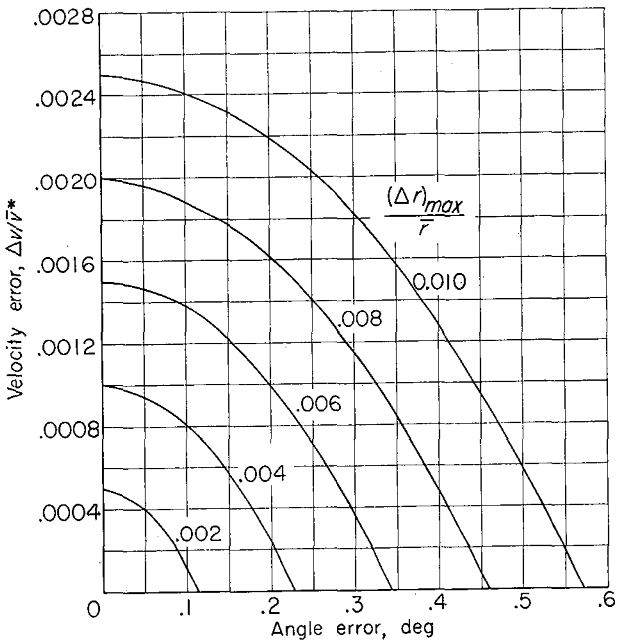


FIGURE 16.—Maximum deviation from circular orbit caused by velocity or angle error.

ORBIT CORRECTION

Under some conditions it may be desired to correct an initially elliptic orbit into a circular orbit. The eccentricity ϵ of the elliptic orbit is assumed to be known and an orbit correction is made at an altitude H, represented by $\overline{\beta}=1-(\overline{r}/r_A)$. At altitude H, the actual velocity $\overline{v}$ is to be corrected to the circular velocity $\overline{v}^*$ by applying a velocity increment Δv in the direction ω. The geometry, therefore, is as indicated in figure 1, except that Δv is pointed in the opposite direction.

From this geometry, together with equations (2) and (22), the following is obtained:

$$\frac{\Delta v^2}{(v_A^*)^2}=\frac{\overline{v}^2}{(v_A^*)^2}+\frac{(\overline{v}^*)^2}{(v_A^*)^2}-\frac{2r_A}{\overline{r}}\frac{\overline{v}^*}{v_A^*}\frac{v_A}{v_A^*}$$

and

$$\cos\omega=\frac{v_A^*}{\Delta v}\left(\frac{r_A}{\overline{r}}\frac{v_A}{v_A^*}-\sqrt{\frac{r_A}{\overline{r}}}\right)$$

In terms of the eccentricity ϵ and the altitude function $\overline{\beta}$, these expressions become

$$\frac{\Delta v}{v_A^*}=\frac{1}{2}\sqrt{\epsilon^2-3\overline{\beta}^2+6\epsilon\overline{\beta}} \tag{71}$$

and

$$\cos\omega=\frac{\beta-\epsilon}{\sqrt{\epsilon^2-3\overline{\beta}^2+6\epsilon\overline{\beta}}} \tag{72}$$

The required velocity increment minimizes when $\overline{\beta}=0$ or 2ϵ, corresponding to the apogee and perigee, respectively; it maximizes when $\overline{\beta}=\epsilon$, corresponding to $\theta=90°$.

CONCLUDING REMARKS

The analysis presented in this report applies for transfer orbits with an eccentricity small compared with unity and for a change in altitude small compared with the distance from the earth's center. Motion about a spherical, nonrotating earth is assumed. Relatively simple expressions for the motion of a descending satellite vehicle and for orbit deviations caused by initial errors were derived.

Errors in the magnitude and direction of the velocity increment used to initiate the descent will contribute to a miss distance in two ways; these errors not only alter the location of atmospheric entry, but also affect the entry angle, which in turn alters the distance traveled within the atmosphere.

The largest single contributing factor towards a miss distance is the misalinement (in the orbital plane) of the retrovelocity increment. If this increment is pointed in direct opposition to the flight path ($\omega=180°$), a 1° misalinement results in a miss distance of 34.5 miles. However, it was shown that this miss distance can be minimized by applying the velocity increment between 120° and 150° below the flight-path direction.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, *October 7, 1958*

APPENDIX—DISCUSSION OF LINEARIZATION PROCEDURE

The two basic assumptions made in the analysis are that the eccentricity of the descent trajectory is small compared with unity and that the change in distance from the center of mass is small. Together, these assumptions imply that the descent trajectory is nearly circular about the center of mass. By assumption, therefore,

$$\left.\begin{array}{l}\epsilon<<1\\ \alpha<<1\end{array}\right\} \tag{A1}$$

The linearization procedure is valid only if all small quantities are of the same order of magnitude. Because the analysis is limited to nearly circular descent trajectories, all radii are by definition of the same order of magnitude. The radii r, $\overline{r}$, and r_A are of nearly equal magnitude, so that $\beta\left(\beta\equiv1-\frac{r}{r_A}\right)$ and $\overline{\beta}\left(\overline{\beta}\equiv1-\frac{\overline{r}}{r_A}\right)$ are both of order $\alpha\left(\alpha\equiv1-\frac{r}{\overline{r}}\right)$. Subsequent expressions will show that $\Delta v/\overline{v}^*$, Φ, and ϵ are also of the same order of magnitude as α.

In deriving the equations presented in this report, all expressions were initially written either exactly, or at least to order ϵ^2. This was done to ensure correctness of the final results to order ϵ. In general, only terms of order ϵ were retained in the resulting expressions.

The derivation of the expression for ϵ, in terms of γ and ω, will now be presented as a typical example. From geometric considerations (fig. 1),

the velocity at the point of orbit deflection is

$$\bar{v}^2=(\bar{v}^*)^2+(\Delta v)^2+2\bar{v}^*\,\Delta v\cos\omega \tag{A2}$$

or

$$\frac{\bar{v}^2}{(\bar{v}^*)^2}=1+2\gamma\cos\omega+\gamma^2 \tag{A3}$$

where

$$\gamma\equiv\Delta v/\bar{v}^* \tag{A4}$$

From equations (5) and (A3), the total energy of the orbit is

$$\frac{2E}{(\bar{v}^*)^2}=-1+2\gamma\cos\omega+\gamma^2 \tag{A5}$$

and the angular momentum becomes

$$\frac{h}{\bar{r}\bar{v}^*}=1+\gamma\cos\omega \tag{A6}$$

The eccentricity is derived from equations (3), (4), (A5), and (A6):

$$\begin{aligned}\epsilon^2&=1+\frac{2Eh^2}{\mu^2}\\&=1+\frac{2E}{(\bar{v}^*)^2}\frac{(\bar{v}^*)^2}{\bar{r}^2(\bar{v}^*)^4}\bar{r}^2(\bar{v}^*)^2(1+\gamma\cos\omega)^2\\&=1+(-1+2\gamma\cos\omega+\gamma^2)(1+2\gamma\cos\omega+\gamma^2\cos^2\omega)\\&=\gamma^2(1+3\cos^2\omega)+2\gamma^3\cos\omega(1+\cos^2\omega)+\gamma^4\cos^2\omega\end{aligned} \tag{A7}$$

and

$$\epsilon=\gamma\sqrt{1+3\cos^2\omega}\left\{1+\gamma\left[\frac{2\cos\omega(1+\cos^2\omega)}{1+3\cos^2\omega}\right]+\gamma^2\frac{\cos^2\omega}{1+3\cos^2\omega}\right\}^{1/2} \tag{A8}$$

Equation (A8) shows that γ is of order ϵ. This derivation also shows the importance of initially retaining terms of order ϵ^2; for if the γ^2 term had been dropped in equation (A3), then ϵ would have been identically equal to zero.

In the text, only the leading term in the expression for ϵ is retained. In this case, therefore, a term of order $\epsilon^{3/2}$ was dropped, while ϵ^1 was not neglected. In general, however, the neglected terms were of order ϵ^2 or higher. In the expressions for error coefficients, only the lowest order terms were retained.

An examination of equations (24) and (25) shows that both $\bar{\beta}$ and β are of the same order of magnitude as γ and, therefore, of order ϵ. Further, because β is of order α, α and ϵ are of the same order of magnitude. Equation (33) shows that Φ is also of order ϵ.

The magnitude of the error introduced by the linearization process cannot easily be ascertained in general terms. However, a typical example may serve to illustrate that the error will be small for many cases of practical interest. Initial conditions at an altitude of 150 statute miles and final conditions at a 50-mile altitude are considered. A velocity increment of 3 percent of satellite velocity, directed at $\omega=180°$, was selected for this example. Results for both the exact and approximate solutions are given in the following table:

Quantity	Exact solution	Present solution	Percent error
Velocity, $v/\bar{v}^*$	0.99493	0.99434	0.06
Angular range, θ, deg	52.471	52.497	0.05
Entry angle, Φ, deg	2.827	2.765	2.22

Thus the errors in velocity and range are seen to be negligible, while the error in entry angle is very small.

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TABLE I.—DESCENT FROM CIRCULAR ORBIT

Quantity	General expression	$\omega=180°$	$\omega=90°, 270°$
Eccentricity, ϵ	$\gamma\sqrt{1+3\cos^2\omega}$	2γ	γ
Altitude function, $\bar{\beta}$	$\gamma(2\cos\omega+\sqrt{1+3\cos^2\omega})$	0	γ
Altitude function, β	$\alpha+\gamma(2\cos\omega+\sqrt{1+3\cos^2\omega})$	α	$\alpha+\gamma$
Entry angle, Φ	$\sqrt{\gamma^2\sin^2\omega-\alpha^2-4\alpha\gamma\cos\omega}$	$\sqrt{4\alpha\gamma-\alpha^2}$	$\sqrt{\gamma^2-\alpha^2}$
Angular range function, $\cos\theta$	$-\frac{2\cos\omega+\alpha/\gamma}{\sqrt{1+3\cos^2\omega}}[1+\alpha+\gamma(2\cos\omega+\sqrt{1+3\cos^2\omega})]$	$\left(1-\frac{\alpha}{2\gamma}\right)(1+\alpha)$	$-\frac{\alpha}{\gamma}(1+\alpha+\gamma)$
Apogee shift function, $\cos\bar{\theta}$	$\frac{-2\cos\omega}{\sqrt{1+3\cos^2\omega}}[1+\gamma(2\cos\omega+\sqrt{1+3\cos^2\omega})]$	1	0
Apogee shift, $\bar{\theta}$		0°	90° when $\omega=90°$ $-90°$ when $\omega=270°$
Velocity ratio, $v/\bar{v}^*$	$1+\alpha+\gamma\cos\omega$	$1+\alpha-\gamma$	$1+\alpha$
Time of flight, $\frac{\bar{v}^*t}{\bar{r}}$	$(1+3\gamma\cos\omega)\left[\cos^{-1}\left(\frac{-\frac{\alpha}{\gamma}-2\cos\omega}{\sqrt{1+3\cos^2\omega}}\right)\mp\cos^{-1}\left(\frac{-2\cos\omega}{\sqrt{1+3\cos^2\omega}}\right)\right]$ $+\sqrt{\gamma^2\sin^2\omega-\alpha^2-4\alpha\gamma\cos\omega}\mp\gamma\sin\omega$ $-$ for $\omega<180°$ $+$ for $\omega>180°$	$(1-3\gamma)\cos^{-1}\left(1-\frac{\alpha}{2\gamma}\right)$ $+\sqrt{4\alpha\gamma-\alpha^2}$	$\cos^{-1}\left(-\frac{\alpha}{\gamma}\right)+\sqrt{\gamma^2-\alpha^2}$ $-\gamma\mp\frac{\pi}{2}$

TABLE II.—REENTRY ERRORS

Quantity	General expression	$\omega=180°$	$\omega=90°, 270°$
Entry angle error, $\bar{v}^*\dfrac{\partial\Phi}{\partial\Delta v}$	$\dfrac{\gamma}{\Phi}\sin^2\omega-\dfrac{2\alpha}{\Phi}\cos\omega$	$\dfrac{2}{\sqrt{\dfrac{4\gamma}{\alpha}-1}}$	$\dfrac{\gamma}{\sqrt{\gamma^2-\alpha^2}}$
Entry angle error, $\dfrac{\partial\Phi}{\partial\omega}$	$\dfrac{\gamma\sin\omega}{\Phi}(\gamma\cos\omega+2\alpha)$	0	$\dfrac{\pm 2\alpha\gamma}{\sqrt{\gamma^2-\alpha^2}}$ + for $\omega<180°$
Range error, $\dfrac{\bar{v}^*}{r}\dfrac{\partial S}{\partial\Delta v}$	$\dfrac{-\alpha}{\Phi\gamma}$	$\dfrac{-1}{\gamma\sqrt{\dfrac{4\gamma}{\alpha}-1}}$	$\dfrac{-\alpha}{\gamma\sqrt{\gamma^2-\alpha^2}}$
Range error, $\dfrac{1}{r}\dfrac{\partial S}{\partial\omega}$	$\dfrac{2}{1+3\cos^2\omega}\left[\left(\dfrac{3}{2}\alpha\cos\omega-\gamma\right)\dfrac{\sin\omega}{\Phi}+1\right]$	$\dfrac{1}{2}$	$2\left(1\mp\dfrac{\gamma}{\sqrt{\gamma^2-\alpha^2}}\right)$ − for $\omega<180°$
Azimuth error, $\dfrac{1}{r}\dfrac{\partial N}{\partial\psi}$	$\gamma\sin(\theta\mp\bar{\theta})$	$\gamma\sin\theta$	α
Time of flight error, $\dfrac{(\bar{v}^*)^2}{\bar{r}}\dfrac{\partial t}{\partial\Delta v}$	$\dfrac{-\alpha}{\Phi\gamma}$	$\dfrac{-1}{\gamma\sqrt{\dfrac{4\gamma}{\alpha}-1}}$	$\dfrac{-\alpha}{\gamma\sqrt{\gamma^2-\alpha^2}}$
Time of flight error, $\dfrac{\bar{v}^*}{\bar{r}}\dfrac{\partial t}{\partial\omega}$	$\dfrac{2}{1+3\cos^2\omega}\left[\left(\dfrac{3}{2}\alpha\cos\omega-\gamma\right)\dfrac{\sin\omega}{\Phi}+1\right]$	$\dfrac{1}{2}$	$2\left(1\mp\dfrac{\gamma}{\sqrt{\gamma^2-\alpha^2}}\right)$ − for $\omega<180°$

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